NASA TECHNICAL NOTE

NASA TN D-2262

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SIMILAR SOLUTIONS OF THE BOUNDARY LAYER EQUATIONS FOR PURELY VISCOUS NON-NEWTONIAN FLUIDS

by C. Sinclair Wells, Jr.

Prepared under Contract No. NASw-~~192~~ 729 *by*
LING-TEMCO-VOUGHT, INC.
Dallas, Texas
for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1964

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April 1964

Completed 15 Jul 64

The contract number which appears on the cover and title page of TN D-2262 is in error, and correctly reads;

Prepared under Contract No. NASw-729 by
LING-TEMCO-VOUGHT, INC.
Dallas, Texas

NASA-Langley, 1964

Issue date: 7-6-64

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Prepared under Contract No. NASw-~~[illegible]~~ by

LING-TEMCO-VOUGHT, INC.

Dallas, Texas

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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Washington, D.C. 20230 -- Price $1.00

SIMILAR SOLUTIONS OF THE BOUNDARY LAYER EQUATIONS FOR PURELY VISCOUS NON-NEWTONIAN FLUIDS

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SUMMARY

The boundary layer equations for flows of purely viscous non-Newtonian fluids are investigated for the purpose of obtaining all possible conditions for which similar solutions exist. Both steady and unsteady flows are investigated. The momentum transport model is the Ostwald-de Waele (power law) formulation which predicts shear-thinning, shear-thickening, and the special case of Newtonian fluids. The similar solutions obtained represent, in all cases, generalizations of boundary layer flows of Newtonian fluids. The solutions are discussed with respect to application to physical flows. Some of the more familiar flows represented are: steady flow over a flat plate, a wedge, and a stagnation region; steady flow in a convergent or divergent channel; and impulsively started flow over an infinite flat plate and a semi-infinite flat plate.

INTRODUCTION

A large portion of boundary layer theory for Newtonian fluids is based on exact solutions which are characterized by affine (linearly transformed in terms of the normal coordinate) boundary layer velocity profiles. These are generally called similar solutions. The practical value of such solutions depends on whether they represent or approximate physical body shapes and flows. Although most of these solutions were obtained by a direct technique for a particular situation, it seems worthwhile to mathematically establish the finite number of conditions which will enable similar solutions to be obtained. After this has been done, the conditions required to produce similar solutions can be investigated with respect to their approximation of physical flows and body shapes, and numerical solutions can be obtained for those cases of physical interest. These conditions for two-dimensional, laminar, boundary layer flows of Newtonian fluids have been recently investigated by Fenter (reference 1). It is the purpose of this paper to apply this technique to establishing and investigating all possible conditions for similar solutions for a useful class of non-Newtonian fluids.

The technique is that of treating what Fenter calls the inverse problem; that is, manipulating the general boundary layer equations to find the conditions under which mathematical similarity exists. The existence of similarity, defined as the existence of affine velocity profiles, implies that the fluid velocity and the spatial and temporal coordinates can be transformed so that the problem is reduced to one involving a single independent variable. The partial differential equation for the stream function then reduces to an ordinary differential equation which can be solved readily. An historical example of a treatment of the inverse problem is that by Falkner and Skan (reference 2) where a family of steady flow conditions producing similarity was found. This investigation can be contrasted with two examples of the direct approach: (1) the Blasius solution for the steady boundary flow on a flat plate (reference 3) (the Blasius solution is also a special case of the Falkner-Skan family of solutions), and (2) the classical Rayleigh problem for unsteady boundary layer flow on an infinite flat plate started impulsively from rest (reference 4).

SYMBOLS

a	constant of proportionality (defined by equation 4)
f	transformed stream function
g	x-dependent factor in inviscid velocity (defined by equation 12)
I, I_2, I_3	Invariants of $\overset{\Rightarrow}{\Delta}$
L	characteristic body length
n	non-Newtonian flow index (defined by equation 4)
p	isotropic static pressure
R_{o_n}	Reynolds number, $\dfrac{\rho\, U_o^{2-n} L^n}{a}$
u	local velocity in the x-direction
U	velocity at the edge of the boundary layer
U_o	reference velocity
U_∞	t-dependent factor in inviscid velocity (defined by equation 12)
v	local velocity in the y-direction
x	coordinate along the surface
y	coordinate normal to the surface
t	time

$\bar{\bar{\tau}}$ shear stress

$\bar{\bar{\Delta}}$ rate of deformation tensor

μ viscosity (defined by equation 1)

μ_e effective viscosity (defined by equation 2)

ρ density

ψ stream function

η similarity variable (defined by equation 15)

ξ function of x and t (defined by equation 15)

FLUID MODEL

In order to describe any fluid, it must be possible to write its stress tensor, $\bar{\bar{\tau}}$, as a function of known variables[1]. For Newtonian fluids, this is easily done:

$$\bar{\bar{\tau}} = \mu\bar{\bar{\Delta}} , \tag{1}$$

where $\bar{\bar{\Delta}}$ is the "rate of deformation tensor" with cartesian components

$$\Delta_{ij} = (\partial v_i/\partial x_j) + (\partial v_j/\partial x_i)$$

and μ is the coefficient of viscosity.

The coefficient of viscosity depends on the local temperature and pressure but not on $\bar{\bar{\tau}}$ or $\bar{\bar{\Delta}}$.

The formulation of $\bar{\bar{\tau}}$ for non-Newtonian fluids is, in general, a complex problem and awaits a well-developed theoretical basis. The description of non-Newtonian fluids is also hampered by the lack of experimental techniques to evaluate various analytical models. Two examples of general fluid models which are based on theoretical considerations but lack sufficient experimental information to relate them to real fluids are: (1) a three-constant model suggested by Oldroyd (reference 5) for non-Newtonian fluids which exhibit elasticity, and (2) a model for non-Newtonian fluids that do not exhibit elasticity which features two scalar functions of $\bar{\bar{\Delta}}$ - the effective viscosity and the "cross viscosity" (references 6, 7).

[1] A discussion of the notation and operations used for second-order tensors is given in reference 8, pp. 726-731.

Lacking a usable model based on theoretical considerations, it is necessary to formulate phenomenological models which can be used in the equations of motion. These models must, however, obey the laws of tensor transformation. Keeping this in mind, it is possible to relate $\bar{\bar{\tau}}$ and $\bar{\bar{\Delta}}$ by means of an effective viscosity (reference 8), for fluids which are purely viscous (no elasticity or anisotropic normal stresses), by

$$\bar{\bar{\tau}} = \mu_e \bar{\bar{\Delta}}, \tag{2}$$

where μ_e, a scalar, is a function of $\bar{\bar{\Delta}}$ as well as a function of temperature and pressure. In order for μ_e to be a scalar function of the tensor $\bar{\bar{\Delta}}$ it must depend only on the invariants of $\bar{\bar{\Delta}}$, which are,

$$I_1 = \Sigma_i \, \Delta_{ii}$$

$$I_2 = \Sigma_i \, \Sigma_j \, \Delta_{ij} \, \Delta_{ji}$$

$$I_3 = \Sigma_i \, \Sigma_j \, \Sigma_k \, \epsilon_{ijk} \, \Delta_{1i} \, \Delta_{2j} \, \Delta_{3k}.$$

I_1 can be shown to be equal to zero for an incompressible fluid and I_3 is identically zero for a two-dimensional flow. Therefore, for two dimensional flow of an incompressible non-Newtonian fluid characterized by equation (2), the stress tensor is given by

$$\bar{\bar{\tau}} = \mu_e(I_2) \, \bar{\bar{\Delta}} \, . \tag{3}$$

At this point several choices of empirical relations for I_2 are available. Due to its convenient form and the fact that it represents a large number of fluids very well, the Ostwald-de Waele (power-law) model is chosen:

$$\mu_e(I_2) = a \left| \sqrt{\tfrac{1}{2} \, \bar{\bar{\Delta}}:\bar{\bar{\Delta}}} \right|^{n-1} \quad (n \geq 0), \tag{4}$$

where $\bar{\bar{\Delta}}:\bar{\bar{\Delta}}$ denotes the scalar product of two tensors. Equation (4) can be combined with equation (3) to give an expression for the shear stress to be considered in the boundary layer equations:

$$\tau_{yx} = \left\{ a \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right\}, \tag{5}$$

where τ_{yx} is the shear stress in the x-direction due to a velocity gradient in the y-direction. If, in addition, the velocity gradient is always positive, as for boundary layer flows, equation (5) can be rewritten as

$$\tau_{yx} = a \left(\frac{\partial u}{\partial y} \right)^n. \tag{6}$$

For values of n less than one, the flow can be described as shear-thinning; and for n greater than one, the flow is shear-thickening. The ranges of values of n are also known, in a less physically descriptive sense, as pseudoplastic and dilatant, respectively. For $n = 1$ the expression describes a Newtonian fluid with $a = \mu$.

Use of the power-law relation given by equation (6) has become well accepted as descriptive of many real fluids. In particular, Schowalter (reference 9) and Acrivos (reference 10) have used the power-law relation to develop similar solutions for steady, two dimensional boundary layer flows which will be shown to be special cases of the general analyses to follow (see Solutions #1, 2, and 3).

ANALYSIS

As mentioned above, the procedure will be to normalize the fluid velocity and the temporal and spatial coordinates and, using the classical boundary layer equations, to write the partial differential equation for the stream function. The problem then will be that of determining the form of a single variable which will allow the transformation of the partial differential equation for the stream function into an ordinary differential equation in terms of a new stream function and the single variable. Once this equation and the appropriate boundary conditions are written, and the significance of the individual terms noted, it will be possible to systematically consider the various conditions which must be satisfied to obtain similar solutions.

The system of equations considered are those credited to Prandtl and must therefore be accompanied by his postulation of a thin boundary layer. For an incompressible, laminar, two-dimensional, thin boundary layer - without regard to the form of the momentum flux term - the continuity and momentum equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \qquad (7)$$

$$\frac{\partial p}{\partial y} = 0,$$

where x and y are coordinates along and normal to the surface of the body, respectively. These coordinates are curvilinear, in general, but owing to the thin boundary layer assumption the equations can be written in their rectilinear forms. The velocities along and normal to the surface of the body are given by u and v, respectively. By substituting equation (6) for the momentum flux, the basic equations are written wholly in terms of gradients to give

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\left\{\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[a\left(\frac{\partial u}{\partial y}\right)^n\right]\right\} \qquad (8)$$

$$\frac{\partial p}{\partial y} = 0 \,.$$

At this point it is convenient to make the variables non-dimensional with the relations,

$$\bar{x} = \frac{x}{L}\,, \quad \bar{y} = \frac{y}{L}, \quad \bar{t} = \frac{tU_o}{L}\,, \quad \bar{u} = \frac{u}{U_o}\,, \quad \bar{v} = \frac{v}{U_o}\,, \quad \bar{U} = \frac{U}{U_o}\,,$$

where L is a characteristic length of the body and U_o is a constant reference velocity. The momentum equation for the x-direction also can be made more convenient by applying the unsteady Bernoulli equation to the outer edge of the boundary layer, as well as applying the momentum equation in the y-direction,

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial U}{\partial t} - U\frac{\partial U}{\partial x}\,,$$

where U is the velocity at the outer edge of the boundary layer. The substitutions give

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial \bar{U}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{U}}{\partial \bar{x}} + \frac{n}{R_{o_n}}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^{n-1}\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right), \qquad (9)$$

where

$$R_{o_n} = \frac{\rho\, U_o^{2-n}\, L^n}{a}\,.$$

The dimensionless stream function, which satisfies the continuity equation, now can be defined as

$$\bar{u} = \frac{\partial \bar{\psi}}{\partial \bar{y}} \qquad \bar{v} = -\frac{\partial \bar{\psi}}{\partial \bar{x}}\,,$$

where $\bar{\psi}$ is related to the ordinary stream function by

$$\bar{\psi} = \frac{\psi}{U_o L}\,.$$

Equation (9) then becomes

$$\bar{\Psi}_{\bar{y}\bar{t}} + \bar{\Psi}_{\bar{y}} \bar{\Psi}_{\bar{y}\bar{x}} - \bar{\Psi}_{\bar{x}} \bar{\Psi}_{\bar{y}\bar{y}} = \frac{\partial \bar{U}}{\partial \bar{t}} + \bar{U} \frac{\partial \bar{U}}{\partial \bar{x}} + \frac{n}{R_o} (\bar{\Psi}_{\bar{y}\bar{y}})^{n-1} \bar{\Psi}_{\bar{y}\bar{y}\bar{y}}, \tag{10}$$

where each subscript indicates partial differentation with respect to that variable.

Three boundary conditions common to all boundary layer problems can be written immediately,

$$\begin{aligned} \bar{\Psi} = \bar{\Psi}_{\bar{y}} &= 0 \quad \text{at} \quad \bar{y} = 0 \\ \bar{\Psi}_{\bar{y}} &\rightarrow \bar{U} \quad \text{as} \quad \bar{y} \rightarrow \infty \,. \end{aligned} \tag{11}$$

Other boundary conditions needed to obtain a solution for a particular physical situation will be determined by the physical situation and will be discussed individually later.

Thus, the first step in the plan to obtain all possible solutions has been accomplished. Equations (10) and (11) give the differential equation and boundary conditions which define the problem in its most general form.

The analysis can be simplified if the velocity at the edge of the boundary layer can be considered to have the form,

$$\bar{U}(\bar{x}, \bar{t}) = \bar{U}_{\infty}(\bar{t}) \; g(\bar{x}) \,, \tag{12}$$

which is valid if the inviscid velocity field is irrotational. Irrotationality is proved if at some time each element of fluid is moving in an irrotational manner. Since only the case of a spatially uniform velocity field at large distances from the body will be considered, the entire inviscid flow field may be considered irrotational.

The next step is to transform the stream function, $\bar{\Psi}$, which is a function of $\bar{x}$, $\bar{y}$ and $\bar{t}$, into a function of a single variable, η , which is in turn a function of $\bar{x}$, $\bar{y}$ and $\bar{t}$. Determination of the form of η , consistent with the condition that η be linearly dependent on $\bar{y}$, will establish the conditions for similarity. f is defined such that

$$f' = \frac{\bar{u}}{\bar{U}} = \frac{1}{\bar{U}} \frac{\partial \bar{\Psi}}{\partial \bar{y}} \,,$$

where the prime indicates differentiation with respect to η.

The boundary conditions given by equations(11) then become

$$f = f' = 0 \quad \text{at} \quad \bar{y} = 0$$

$$f' \longrightarrow 1 \qquad \text{as} \quad \bar{y} \longrightarrow \infty.$$

In order to develop the correct form of η, the relation between the old and new stream function and η must be found. It can be seen that

$$\frac{\partial \bar{\Psi}}{\partial \bar{y}} = \bar{U} f' = \bar{U}_\infty(\bar{t})\, g(\bar{x})\, f'. \tag{13}$$

If, as was stated previously, only forms of η which are linearly dependent on $\bar{y}$ are considered, then

$$\frac{\partial^2 \eta}{\partial \bar{y}^2} = 0.$$

Equation (13) can be integrated to give

$$f(\eta) = \frac{\bar{\Psi}(\bar{x},\bar{y})}{U_\infty(\bar{t}) g(\bar{x})} \frac{\partial \eta}{\partial \bar{y}}. \tag{14}$$

The requirements placed on η are: (1) it must be proportional to y in order to satisfy the first boundary condition given by equation (11); (2) it must involve R_{O_n} such that equation (10) becomes independent of the Reynolds number; and (3) it must be a function of $\bar{x}$ and $\bar{t}$. It is found that making η proportional to $\frac{1}{(R_{O_n})^{n+1}}$ eliminates the Reynolds number dependency, and the introduction of a general function, $\xi(\bar{x}, \bar{t})$, will be useful. Thus the form of η is given by,

$$\eta = \bar{y} \frac{(R_{O_n})^{\frac{1}{n+1}}}{\xi(\bar{x},\bar{t})}. \tag{15}$$

The combination of equations (10),(12),(14) and (15) yields first for each term in equation (10):

$$\bar{\Psi}_{\bar{y}\bar{x}} = \bar{U}_\infty g' f' - \bar{U}_\infty g \eta \frac{\xi_{\bar{x}}}{\xi} f''$$

$$\bar{\Psi}_{\bar{y}\bar{t}} = \bar{U}_\infty' g f' - \bar{U}_\infty g \eta \frac{\xi_{\bar{t}}}{\xi} f''$$

$$\bar{\Psi}_{\bar{x}} = \frac{1}{(R_{O_n})^{\frac{1}{n+1}}} \left[\bar{U}_\infty g \xi_{\bar{x}} f + \bar{U}_\infty g' \xi f + \bar{U}_\infty g \xi_{\bar{x}} \eta f' \right]$$

$$\bar{\psi}_{\bar{y}\bar{y}} = (R_{O_n})^{\frac{1}{n+1}} \frac{\bar{U}_\infty g}{\xi} f''$$

$$\frac{\partial \bar{U}}{\partial \bar{t}} = \bar{U}'_\infty g$$

$$\frac{\partial \bar{U}}{\partial \bar{x}} = \bar{U}_\infty g'$$

$$\bar{\psi}_{\bar{y}\bar{y}\bar{y}} = (R_{O_n})^{\frac{2}{n+1}} \frac{U_\infty g}{\xi^2} f''' ,$$

where primes indicate differentiation with respect to the applicable variable; which in turn yields:

$$\underset{(1)}{f'''} + \left[\xi^{n+1} \frac{(\bar{U}_\infty)^{2-n}}{\eta} g^{2-n} \left(\frac{\xi_{\bar{x}}}{\xi} + \frac{g'}{g} \right) \right] \underset{(2)}{f} (f'')^{2-n}$$

$$- \left[(\bar{U}_\infty)^{2-n} g^{1-n} g' \frac{\xi^{n+1}}{n} \right] \underset{(2)\ (3)}{\frac{f'f'-1}{(f'')^{n-1}}} = \left[\frac{\bar{U}_\infty}{(\bar{U}_\infty)^n} g^{1-n} \frac{\xi^{n+1}}{n} \right] \underset{(5)\ (4)}{\frac{f'-1}{(f'')^{n-1}}} \qquad (16)$$

$$- \left[\frac{(\bar{U}_\infty)^{1-n}}{n} g^{1-n} \xi_{\bar{t}}\ \xi^n \right] \underset{(5)}{\eta (f'')^{2-n}} .$$

As for Newtonian fluids, each term can be identified according to its physical significance by the number under it:

(1) viscous shear stress term
(2) convective terms
(3) pressure gradient due to body shape
(4) pressure gradient due to acceleration of the free-stream
(5) inertia term due to acceleration of the boundary layer fluid

Equation (16) indicates that similarity exists if, and only if, the bracketed factors are either constants or functions of η. Since the terms in the brackets do not contain $\bar{y}$ they cannot be functions of η and must therefore be constants.

The complete ordinary differential boundary layer equation and accompanying general boundary conditions can be written:

$$f''' + A f(f'')^{2-n} - B(f'f'-1)(f'')^{1-n} = C(f'-1)(f'')^{1-n} - D\eta(f'')^{2-n}$$

$$f = f' = 0 \quad \text{when} \quad \eta = 0 \tag{17}$$

$$f' = 1 \quad \text{as} \quad \eta \longrightarrow \infty ,$$

where A, B, C, and D are constants. The problem is now reduced to determining the conditions for which certain body shapes, g, and velocity time histories, U_o, satisfy the following equations:

$$A = \frac{\xi^{n+1}}{n} (\bar{U}_\infty)^{2-n} g^{2-n} \left(\frac{\xi_{\bar{x}}}{\xi} + \frac{g'}{g}\right)$$

$$B = (\bar{U}_\infty)^{2-n} g^{1-n} g' \frac{\xi^{n+1}}{n}$$

$$C = \frac{\bar{U}'_\infty}{(\bar{U}_\infty)^n} g^{1-n} \frac{\xi^{n+1}}{n} \tag{18}$$

$$D = (\bar{U}_\infty)^{1-n} g^{1-n} \frac{\xi_{\bar{t}}\, \xi^n}{n} .$$

The four relationships expressed in equation (18) are sufficient to determine $\xi(\bar{x}\bar{t})$, since $\bar{U}_\infty$ and g represent only two unknowns. As noted in reference 1, the three boundary conditions listed with equation (17) are sufficient to determine the solution since the equation is a third order ordinary differential equation. On the other hand, as many as five boundary conditions were required to solve the equation before it was transformed into similarity variables. It can be seen from equation (18) that the two additional boundary conditions are required by the first-order differentials with respect to $\bar{x}$ and $\bar{t}$.

Treatment of the problem has now progressed to the final state - that of systematically seeking the various solutions to equations (18) - the results of which will provide the conditions and resulting equations which satisfy the similarity assumptions. The investigation will be divided into two categories: steady and unsteady flow.

A. STEADY FLOW:

For steady flow it can be seen that

$$C = D = 0$$

since

$$\bar{U}'_\infty = 0, \text{ or } \bar{U}_\infty = 1$$

and

$$\xi_{\bar{t}} = 0, \quad \text{or} \quad \xi = \xi(\bar{x}) \text{ only.}$$

The conditions for similarity are then given by:

$$A = \frac{\xi^{n+1}}{n} g^{1-n} g' + \frac{g^{2-n}}{n(n+1)} \frac{d\,\xi^{n+1}}{d\,\bar{x}}$$
$$B = g^{1-n} g' \frac{\xi^{n+1}}{n}. \tag{19}$$

Combining the two equations in (19) gives

$$(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A - B\right] = \frac{1}{n}\,\frac{d(\xi^{n+1}\,g^{2-n})}{d\,\bar{x}}\,.$$

Integration yields

$$\frac{\xi^{n+1} g^{2-n}}{n} = (2\,n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]\bar{x} + C_1\,, \tag{20}$$

where C_1 is a constant of integration.

Upon combining equation (20) with the second of equations (19) the relation between g and $\bar{x}$ is found to be, for all steady flows,

$$\frac{g'}{g} = \frac{B}{(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]\bar{x} + C_1}\,. \tag{21}$$

It is seen that there are two classes of solutions to equation (20), depending on whether $\left(\frac{n+1}{2n-1}\right)A-B = 0$ or $\left(\frac{n+1}{2n-1}\right)A-B \neq 0$. The two cases will be investigated separately.

1. $\left(\frac{n+1}{2n-1}\right)A-B \neq 0$

For this case, equation (21) may be written,

$$\frac{dg}{g} = \frac{B\,d\,\bar{x}}{(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]\bar{x} + C_1}\,,$$

which yields, upon integration,

$$g = C_2 \left\{(2\,n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]\bar{x} + C_1\right\}^{\frac{B}{(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]}} , \tag{22}$$

which, when combined with equation (20), gives an expression for ξ:

$$\xi = \frac{n^{\frac{1}{n+1}}\left\{(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]\bar{x} + C_1\right\}^{\frac{A-B}{(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]}}}{C_2^{\frac{2-n}{n+1}}} . \tag{23}$$

From equation (22) it is seen that this solution has an inviscid velocity distribution of the form:

$$\bar{U} = g = \alpha_1\,(\bar{x} + \alpha_2)^m , \tag{24}$$

where the appropriate constants are

$$\alpha_1 = C_2\left\{(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]\right\}^{\frac{B}{(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]}}$$

$$\alpha_2 = \frac{C_1}{(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]} \tag{25}$$

$$m = \frac{B}{(2n-1)\left[\left(\frac{n+1}{2n-1}\right)A-B\right]} .$$

It is seen from equations(25) that α_1 and α_2 can be given any desired values due to the presence of C_1 and C_2. The last of equations (25) also demonstrates that a value for A can be assigned arbitrarily, since B can be varied to give a desired value of m.

It is convenient to let A = 1 in this case. Equations (25) are then solved for B, C_1, and C_2 in terms of the constants in the velocity distribution:

$$B = \frac{m(n+1)}{1+m(2n-1)}$$

$$C_1 = \frac{\alpha_2\ (n+1)}{1+m(2n-1)}$$

$$C_2 = \alpha_1\left[\frac{(n+1)}{1+m(2n-1)}\right]^{-m} .$$

Substitution of these expressions into equations (15), (17), (22) and (23) allows the first differential equation and boundary conditions, together with the appropriate velocity distribution and similarity transformation, to be written. One restriction is made at this point; that is, the singular point

$$m = -\frac{1}{2n-1}$$

is excluded and will be considered as a separate case. Therefore, for the inviscid velocity distribution,

$$\bar{U} = \alpha_1\ (\bar{x} + \alpha_2)^m \qquad \left(m \neq -\frac{1}{2m-1}\right),$$

$$f''' + f(f'')^{2-n} - \left[\frac{m(n+1)}{1+m(2n-1)}\right](f'f'-1)(f'')^{1-n} = 0$$

$$f = f' = 0 \quad \text{at} \quad \eta = 0 \qquad \text{(26) Solution \#1}$$

$$f' \longrightarrow 1 \qquad \text{as } \eta \longrightarrow \infty ,$$

where

$$\eta = \bar{y}\left[\frac{R_{o_n}\ \alpha_1^{2-n}\ [1+m(2n-1)]}{n(n+1)(x+\alpha_2)^{1+m(n-2)}}\right]^{\frac{1}{n+1}} .$$

These equations can be solved numerically to provide similar solutions for this family of flows.

For the case of $m = -\frac{1}{2n-1}$, it can be seen from the third of equations (25) that

$$\left(\frac{n+1}{2n-1}\right)A = 0,$$

which implies that A = 0 and, therefore, that B is arbitrary. Then equation (23) gives an expression for ξ, after substitution and manipulation:

$$\xi = \left[\frac{-B\, n\, (2n-1)}{\alpha_1^{\,2-n}}\right]^{\frac{1}{n-1}} (\bar{x} + \alpha_2)^{\frac{1}{2n-1}} .$$

In order to avoid problems with imaginary numbers, B is chosen to be

$$B = \frac{-\alpha_1^{\,2-n}}{|\alpha_1|^{2-n}} .$$

The velocity distribution for this case is

$$\bar{U} = \frac{\alpha_1}{(\bar{x} + \alpha_2)^{\frac{1}{2n-1}}} ,$$

where

$$B = -\frac{\alpha_1^{\,2-n}}{|\alpha_1|^{2-n}}$$

$$C_1 = \frac{\alpha_1^{\,2-n}\, \alpha_2}{|\alpha_1|^{2-n}} \cdot (2n-1)$$

$$C_2 = \alpha_1 \left[\frac{\alpha_1^{\,2-n}}{\alpha_1^{\,2-n}} (2n-1)\right]^{\frac{1}{2n-1}} .$$

Substituting in equations (15) and (17) gives for flows of the following inviscid velocity distribution:

$$\bar{U} = \frac{\alpha_1}{(\bar{x} + \alpha_2)^{\frac{1}{2n-1}}} ,$$

$$f''' + \frac{\alpha_1^{\,2-n}}{|\alpha_1|^{2-n}} (f'f'-1)(f'')^{1-n} = 0$$

$$f = f' = 0 \quad \text{at} \quad \eta = 0 \qquad (27)$$

Solution #2

$$f' \to 1 \qquad \text{as} \quad \eta \to \infty,$$

where

$$\eta = \bar{y} \left[\frac{R_{o_n} \, |\alpha_1|^{2-n}}{n(2n-1)(\bar{x}+\alpha_2)^{\frac{n-1}{2n-1}}} \right]^{\frac{1}{n+1}} .$$

2. $\underline{\left(\frac{n+1}{2n-1}\right) A-B = 0, \; A = \left(\frac{2n-1}{n+1}\right) B}$

This condition reduces equation (21) to

$$\frac{g'}{g} = \frac{B}{C_1},$$

which can be integrated to give

$$g = C_3 \, e^{\frac{B}{C_1} \bar{x}} . \qquad (28)$$

Combining equations (20) and (28) gives the expression for ξ to be

$$\xi = \left(\frac{n \, C_1}{C_3^{\,2-n}}\right)^{\frac{1}{n+1}} e^{-\left(\frac{2-n}{n+1}\right)\left(\frac{B}{C_1}\right) \bar{x}} . \qquad (29)$$

Equation (28) shows that this solution is valid for velocity distributions of the following type:

$$\bar{U} = \alpha_1 e^{\alpha_2 \bar{x}} , \tag{30}$$

where

$$C_1 = \frac{B}{\alpha_2}$$

$$C = \alpha_1 .$$

Equation (29) can then be written as

$$\xi = \left(\frac{n\ B}{\alpha_2\ \alpha_1^{2-n}}\right)^{\frac{1}{n+1}} e^{-\left(\frac{2-n}{n+1}\right) \alpha_2 \bar{x}} .$$

If α_1 is restricted to values greater than zero and B is chosen to be $\alpha_2/|\alpha_2|$ (since B is arbitrary for the reason previously stated) no generality is lost and the problem of imaginary numbers is avoided. Thus for inviscid velocity distributions given by:

$$\bar{U} = \alpha_1 e^{\alpha_2 \bar{x}},$$

$$f''' + \frac{\alpha_2}{|\alpha_2|}\left[\tfrac{1}{2} f(f'')^{2-n} - (f'f' - 1)(f'')^{1-n}\right] = 0 \qquad \begin{matrix}(31)\\ \text{Solution}\\ \#3\end{matrix}$$

$$f = f' = 0 \qquad \text{at} \quad \eta = 0$$

$$f' \longrightarrow 1 \qquad \text{as} \quad \eta \longrightarrow \infty ,$$

where

$$\eta = \bar{y}\left[\frac{R_{o_n}\ \alpha_2\ \ \alpha_1^{2-n}\ \ e^{(2-n)\alpha_2 \bar{x}}}{n}\right]^{\frac{1}{n+1}} .$$

B. UNSTEADY FLOW

It is convenient to divide the investigation of similarity for time-dependent flows into two cases; flows with streamwise velocity gradients, and flows without streamwise velocity gradients.

1. Unsteady Flow with Streamwise Velocity Gradients

It can be seen that this choice leads to the conclusion that all of the constants defined by equations (16) are non-zero, since g' as well as $\xi_{\bar{t}}$, is non-zero. Upon combining the second and third of equations (18), it is seen that,

$$\frac{\bar{U}_\infty'}{\bar{U}_\infty^2} = \frac{C}{B} g' = C_4 , \tag{32}$$

where C_4 is a constant which will be evaluated later. Integration of the $\bar{U}_\infty$ term of equation (32) gives,

$$\bar{U}_\infty = \frac{1}{C_5 - C_4 \bar{t}} , \tag{33}$$

and, similarly, the g term of equation (32) can be integrated, with the result,

$$g = \frac{C_4 B}{C} \bar{x} + C_6 . \tag{34}$$

Combining equation (34) with the third of equations (18) gives the following expression for ξ:

$$\xi = \left[\frac{C_n}{C_4} (C_5 - C_4 \bar{t})^{2-n} \left(\frac{C_4 B}{C} \bar{x} + C_6 \right)^{n-1} \right]^{\frac{1}{n+1}} . \tag{35}$$

Equations (33) and (34) can be combined to give the inviscid velocity distributions for which this solutions exists:

$$\bar{U} = \frac{\alpha_1 + \alpha_2 \bar{x}}{\alpha_3 - \alpha_4 \bar{t}} ,$$

where

$$\alpha_1 = C_6 \qquad \alpha_3 = C_5$$

$$\alpha_2 = \frac{C_4 B}{C} \qquad \alpha_4 = C_4 \ .$$

If the above relations are substituted in equations (18) the following relations for A, C, and D in terms of B are found to be:

$$A = \frac{2\,n}{n+1} \quad B$$

$$C = \frac{\alpha_4}{\alpha_2} \quad B$$

$$D = \left(\frac{n-2}{n+1}\right) \frac{\alpha_4}{\alpha_2} \quad B \ .$$

B is again arbitrary and is chosen equal to 1. Thus for the following inviscid flow:

$$\bar{U} = \frac{\alpha_1 + \alpha_2 \bar{x}}{\alpha_3 - \alpha_4 \bar{t}} \ , \qquad \text{(36) Solution \#4}$$

$$f''' + \frac{2n}{n+1} f(f'')^{2-n} - (f'f'-1)(f'')^{1-n}$$

$$= \frac{\alpha_4}{\alpha_2} (f'-1)(f'')^{1-n} - \frac{\alpha_4}{\alpha_2} \frac{n-2}{n+1} \eta \, (f'')^{2-n}$$

$$f = f' = 0 \quad \text{at} \quad \eta = 0$$

$$f' \longrightarrow 1 \qquad \text{as} \quad \eta \longrightarrow \infty \ ,$$

where

$$\eta = \bar{y} \left[\frac{\alpha_2 \ R_{On}}{(\alpha_3 - \alpha_4 \bar{t})^{2-n} \ (\alpha_2 \bar{x} + \alpha_1)^{n-1}} \right]^{\frac{1}{n+1}} \ .$$

2. Unsteady Flow without streamwise velocity gradients (flat plates at zero angle of attack)

The case of no streamwise velocity gradients gives $g = 1$ and $g' = 0$, which reduces equations (18) to the following:

$$A = \frac{1}{n(n+1)} \left(\xi^{n+1}\right)_{\bar{x}} \left(\bar{U}_\infty\right)^{2-n}$$

$$B = 0$$

$$C = \frac{1}{n} \frac{\bar{U}_\infty'}{(\bar{U}_\infty)^n} \xi^{n+1}$$

$$D = \frac{(\bar{U}_\infty)^{1-n}}{n(n+1)} \left(\xi^{n+1}\right)_{\bar{t}}.$$

It is possible to consider the time dependency of the boundary layer both with and without a time dependent inviscid flow. This is done by considering two cases; that of $\bar{U}_\infty' = 0$ and $\bar{U}_\infty' \neq 0$.

a. $\bar{U}_\infty' = 0$

For this case, $\bar{U}_\infty = 1$ and $C = 0$. Integrating the expressions for A and D with respect to $\bar{x}$ and $\bar{t}$, respectively,

$$\frac{1}{n(n+1)} \xi^{n+1} = A\,\bar{x} + T(\bar{t}) \tag{37}$$

$$\frac{1}{n(n+1)} \xi^{n+1} = D\,\bar{t} + X(\bar{x}).$$

Equations (37) combine to give:

$$A\,\bar{x} - X(\bar{x}) = D\bar{t} - T(\bar{t}) = C_7 ,$$

or

$$T(\bar{t}) = D\bar{t} - C_7$$

and

$$X(\bar{x}) = A\bar{x} - C_7 .$$

Substitution of these results into either of equations (37) gives an expression for ξ:

$$\xi = \left[n\,(n+1)\,(A\,\bar{x} + D\bar{t} - C_7) \right]^{\frac{1}{n+1}} . \tag{38}$$

Two of the constants in equation (38) will be required to satisfy the two possible additional boundary conditions; therefore, one will be arbitrary. A value of D = 2 will be assigned. It should be noted here that the special case of D = 0 reduces to a special case of Solution #1. Equation (38) can then be written:

$$\xi = \left[n\,(n+1)(2)(\alpha_1\,\bar{x} + \bar{t} + \alpha_1) \right]^{\frac{1}{n+1}} , \tag{39}$$

where

$$\alpha_1 = \frac{A}{2} \qquad \text{and} \qquad \alpha_2 = -\frac{C_7}{2} .$$

Therefore, for inviscid flow fields of the type:

$$\bar{U} = 1,$$

$$f''' + 2\,\alpha_1\,f\,(f'')^{2-n} + 2\eta\,(f'')^{2-n} = 0$$

$$f = f' = 0 \qquad \text{at} \qquad \eta = 0 \tag{40}$$

$$f' \longrightarrow 1 \qquad \text{as} \qquad \eta \longrightarrow \infty ,$$

(40) Solution #5

where

$$\eta = \bar{y} \left[\frac{R_{o_n}}{2n(n+1)(\alpha_1\,\bar{x} + \bar{t} + \alpha_2)} \right]^{\frac{1}{n+1}} .$$

b. $\underline{\bar{U}'_\infty \neq 0}$

Combining the expressions for C and D yields

$$D + (1-n)\,C = \frac{1}{n(1+n)} \frac{d(U_{\infty}^{1-n} \xi^{1+n})}{d\bar{t}},$$

and after integration gives

$$\bar{U}_{\infty}^{1-n} \xi^{n+1} = n(1+n)\left\{ \left[D + (1-n)\,C \right] \bar{t} + C_8 \right\}. \qquad (41)$$

If equation (41) is substituted into the expression for C, an ordinary differential equation for the velocity distribution results:

$$\frac{\bar{U}_{\infty}'}{\bar{U}_{\infty}} = \frac{C}{(n+1)\left\{[D + (1-n)\,C]\bar{t} + C_8\right\}}, \qquad (42)$$

which can be integrated for each of two cases; that of $D + (1-n)\,C = 0$ and $D + (1-n)\,C \neq 0$. If $D + (1-n)\,C \neq 0$,

$$\bar{U}_{\infty} = C_9 \left\{[D + (1-n)\,C]\,\bar{t} + C_8\right\}^{\frac{C}{(1+n)\,[D+(1-n)\,C]}}, \qquad (43)$$

and if $D = 2$ is chosen arbitrarily, the velocity distribution is given in general by:

$$\bar{U} = \alpha_1 (\bar{t} + \alpha_2)^m,$$

where

$$\alpha_1 = C_9 \,[2 + (1-n)C]^m$$

$$\alpha_2 = \frac{C_8}{2 + (1-n)C}$$

$$m = \frac{C}{(1+n)[2+(1-n)C]}.$$

It can be seen that $A = 0$ since the expression for ξ is independent of $\bar{x}$, and A is proportional to $\xi_{\bar{x}}$. This solution can be written, for inviscid flow fields of the type:

$$\bar{U} = \alpha_1 (\bar{t} + \alpha_2)^m ,$$

$$f''' - \frac{2m(n+1)}{1-m(1+n)(1-n)} (f'-1)(f'')^{1-n} + 2 (f'')^{2-n} = 0 \tag{44}$$

Solution #6

$$f = f' = 0 \qquad \text{at} \qquad \eta = 0$$

$$f' \longrightarrow 1 \qquad \text{as} \qquad \eta \longrightarrow \infty ,$$

where

$$\eta = \bar{y} \left[\frac{1-m(1+n)(1-n) \quad \alpha_1^{1-n} \quad R_{On}}{2n(n+1) \; (\bar{t} + \alpha_2)^{1-m \; (1-n)}} \right]^{\frac{1}{n+1}} .$$

If D + (1-n) C = 0, equation (42) can be solved to give

$$\bar{U}_\infty = C_{10} \; e^{\frac{C}{(n+1) \; C_8} \bar{t}} \tag{45}$$

or,

$$\bar{U} = \alpha_1 \, e^{\alpha_2 \bar{t}} ,$$

where

$$\alpha_1 = C_{10} \qquad\qquad \alpha_2 = \frac{C}{(n+1)C_8} .$$

If C is chosen to be $\alpha_2 / |\alpha_2|$, the expression for ξ is written,

$$\xi = \left[\frac{n\alpha_1^{\;n-1}}{|\alpha_2|} \quad (e^{\alpha_2 \bar{t}})^{\;n-1} \right]^{\frac{1}{n+1}} ,$$

which gives, for the inviscid flow field,

$$\bar{U} = \alpha_1 e^{\alpha_2 \bar{t}},$$

$$f''' - \frac{\alpha_2}{|\alpha_2|}(f' - 1)(f'')^{1-n} = 0 \qquad (46)$$

Solution #7

$$f = f' = 0 \qquad \text{at} \quad \eta = 0$$

$$f' \longrightarrow 1 \qquad \text{as} \quad \eta \longrightarrow \infty,$$

where

$$\eta = \bar{y}\left[\frac{R_{o_n} |\alpha_2|}{n\left(\alpha_1 e^{\alpha_2 \bar{t}}\right)^{n-1}}\right]^{\frac{1}{n+1}}.$$

DISCUSSION

A discussion of each of the seven sets of conditions which produce similar boundary layer solutions is now possible. The primary motivation is that of relating the requirements for similarity to physically real flows. In some of the cases the relation is exact; in others the relation is only approximate and the usefulness of application to real flow cases will depend on the cleverness of the user.

Each solution is, of course, a generalization of the Newtonian case. It is interesting to note that the same number of similar solutions exist for power-law fluids as for Newtonian fluids. Detailed investigation of each solution may, however, show restrictions on the range of values of n for which similar solutions exist - a different range than that given by consideration of the boundary layer approximations.

Some unusual limitations of the boundary-layer theory should be noted at this point. The form of R_{o_n},

$$R_{o_n} = \frac{\rho\, U_o^{2-n} L^n}{K},$$

states that, for values of $n < 2$, R_{o_n} can be made sufficiently large by increasing U. If, however, $n > 2$, there is an upper limit on U_o for making R_{o_n} sufficiently large. There is also a lower limit on U_o for $n > 0$ because the power-law formulation is not valid for most fluids at low values of du/dy. Although it appears that there is some range of U_o where the boundary layer approximations are valid for $n > 2$, the limitations probably restrict use of the theory to values of $n < 2$.

As was noted earlier, five boundary conditions are required to define a similar solution, in general. Three boundary conditions were provided initially, as characteristic of all boundary layer problems. The remaining boundary conditions, if required for a particular solution, are found as constants of integration in the expression for ξ. These extra boundary conditions are, therefore, related intimately to the physical system considered and will be discussed as such.

Numerical analysis of each solution will provide unique values of f' for values of η equal to and greater than zero, for selected values of n. This makes possible the definition of a boundary layer thickness, δ, which is defined as the value of $\bar{y}$ when f' has reached an arbitrary percentage of unity. For example, if the value of f' is chosen to be 0.99, then,

$$f'\ (\eta_\delta) = 0.99$$

where η_δ is the value of η at $y = \delta$ and $f' = 0.99$. δ can then be written as,

$$\delta = \frac{\eta_\delta\ \xi(\bar{x},\bar{t})}{(R_{O_n})^{\frac{1}{n+1}}} \ .$$

It should be noted in passing that this expression shows the boundary layer thickness to be inversely proportional to $(R_{O_n})^{\frac{1}{n+1}}$ rather than the more familiar $(R_{O_n})^{\frac{1}{2}}$ for the special case of Newtonian flow.

Solution #1

Equation (26) represents a generalization of a family of solutions for Newtonian fluids first deduced by V. M. Falkner and S. W. Skan (reference 2) and later investigated in detail by D. R. Hartree (reference 11). The several types of inviscid velocity distributions probably have the most useful physical applications of all of the similar solutions for Newtonian fluids. The same application to physical flows is found for non-Newtonian fluids. As was mentioned earlier, Solution #1, as well as Solutions #2 and #3, represent a generalization of the steady flow solutions stated recently by Showalter for power-law fluids.

Since α_2 only serves to translate $\bar{x}$ from the origin, α_2 can be set equal to zero without a loss of generality. This also satisfies the initial boundary condition: $\delta = 0$ when $\bar{x} = 0$ and $\bar{U} = 0$. For $\alpha_1 > 0$ the equation for $\bar{U}$ is a generalization, for power-law fluids, of the Falkner-Skan wedge flows. Discussion of the various inviscid flow fields can be facilitated by rewriting the expression for $\bar{U}$ and the differential equation, together with expressions for η and δ :

$$\bar{U} = \alpha_1\ (\bar{x})^m.$$

$$f''' + f(f'')^{2-n} - \frac{m(n+1)}{1+m(2n-1)} (f'f'-1)(f'')^{1-n} = 0$$

$$\eta = \bar{y} \left[\frac{(R_{o_n}) \, [1+m(2n-1)] \, \alpha_1^{2-n}}{n(n+1)(\bar{x})^{m(n-2)+1}} \right]^{\frac{1}{n+1}}$$

$$\delta = \eta_\delta \left[\frac{n(n+1) \, (\bar{x})^{m(n-2)+1}}{R_{o_n} \, [1+m(2n-1)]^{\alpha_1^{(2-n)}}} \right]^{\frac{1}{n+1}} .$$

The general case of $0 < m < 1$ physically represents the potential flow in the neighborhood of the stagnation point of a wedge. Potential theory (reference 12) gives the relation between the full wedge angle $\pi\beta$, and the power of the velocity distribution, m:

$$\beta = \frac{2\,m}{1+m} .$$

From the previously given definition of the constant B:

$$B = \frac{m(n+1)}{1+m(2n-1)} ,$$

the relation between β and B can be found in terms of m and n;

$$\beta = \frac{2B}{(n+1) + B(2-2n)} .$$

This is seen to reduce to $\beta = B$ for Newtonian fluids.

The flow over a flat plate at zero incidence is obtained if $m = 0$, where $B = 0$ and $\bar{U} = \alpha_1$. This problem has recently been investigated numerically for power-law fluids by Acrivos et. al. (reference 10). It is interesting to note that the boundary layer thickness grows in direct proportion to $(\bar{x})^{\frac{1}{n+1}}$ compared to $\bar{x}^{\frac{1}{2}}$ for Newtonian fluids. Thus, for pseudoplastic fluids $(n < 1)$ δ would grow much faster with $\bar{x}$ than it would for Newtonian fluids, other conditions being equal.

For m = 1 it is seen that $\beta = 1$ and $\bar{U} = \alpha_1 \bar{x}$ which is the physical case of stagnation flow for all values of n. A significant characteristic of this solution is the variation of δ, which is directly proportional to $(\bar{x})^{\frac{n-1}{n+1}}$. Again using the example of pseudoplastic fluids, δ decreases with increasing $\bar{x}$ for $0<n<1$. For $n>1$, δ increases with increasing $\bar{x}$. δ is either infinite (or finite for $\alpha_2 \neq 0$) or zero at $\bar{x} = 0$, depending on whether $0 < n < 1$ or $n > 1$, respectively. Comparatively, the $\bar{x}$- dependence of δ disappears for Newtonian fluids.

The case of m = 1 and $\alpha_2 > 0$ should be mentioned. This results in an inviscid velocity distribution of the type:

$$\bar{U} = a_1 - a_2\bar{x}$$

where $a_1 > a_2\bar{x}$ and $a_2 > 0$. This corresponds physically to flow in a straight channel followed by a divergent channel.

Solution #2

The solution given by equation (27) is also a generalization of one of the Falkner-Skan cases. Again α_2 can be set equal to zero without a loss of generality since the term amounts to a translation of the $\bar{x}$ coordinate:

$$\bar{U} = \frac{\alpha_1}{(\bar{x})^{\frac{1}{2n-1}}}$$

$$f''' + \frac{(\alpha_1)^{2-n}}{|\alpha_1|^{2-n}} (f'f'-1)(f'')^{1-n} = 0$$

$$\eta = \bar{y} \left[\frac{2 R_{o_n} |\alpha_1|^{2-n}}{n(2n-1)(\bar{x})^{\frac{n+1}{2n-1}}}\right]^{\frac{1}{n+1}}$$

$$\delta = \eta_\delta \left[\frac{n(2n-1)(\bar{x})^{\frac{n+1}{2n-1}}}{2 R_{o_n} |\alpha_1|^{2-n}}\right]^{\frac{1}{n+1}} .$$

For Newtonian fluids, the inviscid velocity distribution is that of either a source or a sink, depending on the sign of x_1 and hence $\bar{U}$. No general statement can be made relating the inviscid velocity distributions for Newtonian fluids to potential theory. Practically, the general expression for $\bar{U}$ approximates flow in diverging or converging curved channels. The special case of Newtonian fluids approximates a diverging or converging channel with straight walls. This can be only an approximation because the solution requires that $\bar{U} \to \infty$ and $\delta \to 0$ (for $n < 5$) as $\bar{x} \to 0$ and these conditions are not satisfied in a real channel where the inviscid velocity is finite at $\bar{x} = 0$. When applied to flow in channels the solution becomes more accurate as $\bar{x}$ increases.

Solution #3

Equations (31) describe the generalization of a seldom-discussed Falkner-Skan solution. It can be seen that values of α_1 other than unity translate the $\bar{x}$ origin, therefore no generality is sacrificed if α_1 is set equal to unity and:

$$\bar{U} = e^{\alpha_2 \bar{x}}$$

$$f''' + \frac{\alpha_2}{|\alpha_2|}\left[\frac{1}{2} f(f'')^{2-n} - (f'f'-1)(f'')^{1-n}\right] = 0$$

$$\eta = \bar{y}\left[\frac{R_{o_n} |\alpha_2| e^{(2-n)\alpha_2 \bar{x}}}{n}\right]^{\frac{1}{n+1}}$$

$$\delta = \eta_\delta \left[\frac{n}{R_{o_n}(\alpha_2)\, e^{(2-n)(\alpha_2)\bar{x}}}\right]^{\frac{1}{n+1}}.$$

Any application of this solution to a physical flow must satisfy the initial condition:

$$\delta = \eta_\delta \left[\frac{n}{R_{o_n}(\alpha_2)}\right]^{\frac{1}{n+1}}$$

at $\bar{x} = 0$, where

$$\alpha_2 = \left(\frac{d\bar{U}}{d\bar{x}}\right)_{\bar{x}=0}$$

This can be compared with the case of stagnation flow for Solution #1, in which:

$$\delta = \eta_\delta \left[\frac{(n+1)\ (\bar{x})^{n-1}}{2\ R_{o_n}\ (\alpha_1)^{2-n}} \right]^{\frac{1}{n+1}} ,$$

where

$$\alpha_1 = \left(\frac{d\bar{U}}{d\bar{x}} \right)_{\bar{x}\,=\,0} .$$

For Newtonian fluids, where both cases lack an $\bar{x}$-dependence, there exists the possibility that the two solutions can be matched at some point downstream of the stagnation point. This may also be true for the more general case of power-law fluids; however, further investigation and numerical analysis is required to determine how well the solution represents a physical situation. Certainly such a match of solutions would not produce an exact representation at the origin, but may become sufficiently accurate as $\bar{x}$ increases from the origin.

Solution #4

The solution for unsteady flow given by equations (36) can be rewritten by again letting α_1, the translation of $\bar{x}$, go to zero with no loss of generality:

$$\bar{U} = \frac{\alpha_2 \bar{x}}{\alpha_3 - \alpha_4 \bar{t}}$$

$$f''' + \frac{2n}{n+1} f(f'')^{2-n} - (f'f'-1)(f'')^{1-n}$$

$$= \frac{\alpha_4}{\alpha_2} (f'-1)(f'')^{1-n} - \frac{\alpha_4}{\alpha_2} (\frac{n-2}{n\ 1})\ \eta\ (f'')^{2-n}$$

$$\eta = \bar{y} \left[\frac{\alpha_2\ R_{o_n}}{(\alpha_3 - \alpha_4 \bar{t})^{2-n}\ (\alpha_2 \bar{x})^{n-1}} \right]^{\frac{1}{n+1}}$$

$$\delta = \eta_\delta \left[\frac{(\alpha_3-\alpha_4\bar{t})^{2-n} (\alpha_2\bar{x})^{n-1}}{\alpha_2 R_{o_n}} \right]^{\frac{1}{n+1}} .$$

These equations, for Newtonian fluids, were investigated and reported recently by Yang (reference 12).

It should be noted that δ is a function of both $\bar{t}$ and $\bar{x}$, in general; but it is a function of $\bar{t}$ only for Newtonian fluids. As for the steady flow stagnation solution, δ decreases for increasing $\bar{x}$ for $0<n<1$, and increases for increasing $\bar{x}$ for $n>1$.

The inviscid velocity distribution is that of a stagnation flow with a hyperbolic history. The significance of this type of unsteady flow can be shown by first noting that this type of flow will result when the forces acting on the body are proportional to the square of the velocity:

$$\frac{d\bar{U}_\infty}{d\bar{t}} \propto \bar{U}_\infty^2 .$$

It can be seen that this type of velocity history will result when all the forces acting on the body are forces which obey the simple quadratic resistance law. Specifically, for the case of non-lifting bodies in which the drag force predominates the acceleration is approximately as follows:

$$\frac{d\bar{U}_\infty}{d\bar{t}} = \frac{g\rho}{2\left(\frac{W}{C_D A}\right)} \bar{U}_\infty^2 .$$

Solution #4 will very closely approximate this physical case if the drag coefficient and the density are constant or slowly varying.

Solution #5

Equations (40) give the general solution for a flat plate at zero incidence moving with a constant velocity in its own plane. Since the term associated with inertia due to the accelerating boundary layer fluid is retained, the physical situation must be that of a transient boundary layer trying to adjust to new steady state conditions. The initial condition of $\delta = 0$ for $\bar{x} = \bar{t} = 0$ can be satisfied by setting α_2 equal to zero. The problem can be simplified considerably by eliminating the $\bar{x}$ dependence of η (letting $\alpha_1 = 0$). Since this eliminates the convective term, the solution corresponds physically to the case of an impulsively moved infinite flat plate or a finite plate at the first instant of motion and is, for Newtonian fluids, the classical Rayleigh problem. This can be written:

$$\bar{U} = 1$$

$$f''' + 2\,\eta(f'')^{2-n} = 0$$

$$\eta = \bar{y}\left[\frac{R_{o_n}}{2n\,(n+1)\bar{t}}\right]^{\frac{1}{n+1}}$$

$$\delta = \eta\left[\frac{2n\,(n+1)\bar{t}}{R_{o_n}}\right]^{\frac{1}{n+1}} .$$

The more general case for $\alpha_1 \neq 0$ retains the convective term and therefore corresponds to a finite flat plate, with its leading edge at $\bar{x} = 0$, impulsively put into motion in its own plane. This situation is given by:

$$\bar{U} = 1$$

$$f''' + 2\alpha_1 f(f'')^{2-n} + 2\,\eta\,(f'')^{2-n} = 0$$

$$\eta = \bar{y}\left[\frac{R_{o_n}}{2n(n+1)(\alpha_1\bar{x} + \bar{t})}\right]^{\frac{1}{n+1}}$$

$$\delta = \eta_\delta\left[\frac{2n(n+1)(\alpha_1\bar{x} + \bar{t})}{R_{o_n}}\right]^{\frac{1}{n+1}} .$$

Solution #6

Equations (44) describe an inviscid flow with a power-law velocity history. Since α_2 again corresponds only to a displacement of a coordinate, in this case $\bar{t}$, it can be set equal to zero with no significant loss of generality:

$$\bar{U} = \alpha_1\,(\bar{t})^m \qquad (\bar{t} \geq 0)$$

$$f''' - \frac{2m(n+1)}{1-m(1+n)(1-n)} (f'-1)(f'')^{1-n} + 2\eta(f'')^{2-n} = 0$$

$$\eta = \bar{y}\left[\frac{[1-m(n+1)(1-n)]\ \alpha_1^{\ 1-n} R_{o_n}}{2n(n+1)(\bar{t})^{1-m(1-n)}}\right]^{\frac{1}{n+1}}$$

$$\delta = \eta_\delta \left[\frac{2n(n+1)(\bar{t})^{1-m(1-n)}}{[1-m(n+1)(1-n)]\alpha_1^{\ 1-n} R_{o_n}}\right]^{\frac{1}{n+1}} .$$

Since no convection terms are present, the solution again applies to an infinite flat plate or a finite plate at the first instant of motion.

The initial condition, $\delta = 0$ at $\bar{x} = 0$ is satisfied exactly for infinite plates. It is interesting to note that for $m = 0$, the problem is identical to that of the special case of Solution #5 where α_1 and $\alpha_2 = 0$, the case of the impulsively moved plate at a constant velocity.

Solution #7

The solution described by equations (46) also applies to an infinite flat plate moving in its own plane, since no convection terms are present. The inviscid velocity history is a simple exponential curve and α_1 can be set equal to unity since it corresponds to a displacement of the time scale. The conditions of the solution can be rewritten:

$$\bar{U} = e^{\alpha_2 \bar{t}}$$

$$f''' - \frac{\alpha_2}{|\alpha_2|} (f'-1)(f'')^{1-n} = 0$$

$$\eta = \bar{y}\left[\frac{R_{o_n}|\alpha_2|}{n(e^{\alpha_2 \bar{t}})^{n-1}}\right]^{\frac{1}{n+1}}$$

$$\delta = \eta_\delta \left[\frac{n(e^{\alpha_2 \bar{t}})^{n-1}}{R_{o_n}|\alpha_2|}\right]^{\frac{1}{n+1}} .$$

The boundary layer thickness is seen to be finite at $t = 0$ for all values of n. δ is seen to be constant for Newtonian fluids.

CONCLUSIONS

The following conclusions have been made as a result of investigating the two-dimensional, laminar, incompressible boundary layer equation for purely viscous, power-law non-Newtonian fluids:

1. The solutions found for steady and unsteady flows are generalizations of the solutions found by Fenter for Newtonian fluids. The number of similar solutions is found to be the same as for Newtonian fluids.

2. With regard to the steady flow solutions found, all are generalized versions of the Falkner-Skan flows for Newtonian fluids. Solutions #1, #2, and #3 are generalized versions of similar solutions for power-law fluids investigated by Schowalter.

3. Solution #4 is found to represent a stagnation flow with a family of flow histories. The boundary layer thickness is seen to be a function both of $\bar{x}$ and $\bar{t}$ for the general case, where δ is a function of $\bar{t}$ only for Newtonian fluids.

4. Solution #5 represents the flow over a flat plate at zero incidence, moving at a constant velocity, in which the transient boundary layer is trying to adjust to new steady state conditions. The solution includes as special cases the flows over an infinite and semi-infinite flat plate impulsively put into motion in their own planes.

5. Solution #6 describes a flow with a power-law velocity history over an infinite flat plate.

6. Solution #7 represents flow over an infinite flat plate moving in its own plane. The inviscid velocity history is an exponential function. The boundary layer thickness is found to be a function of time for the general case, where it is constant for Newtonian fluids.

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